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A Tutorial Presentation

BASICS OF HYDROSTATIC SPINDLES THEORETICAL ANALYSIS: ANALYTICAL AND NUMERICAL APPROACHES

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ASPE 2019 TUTORIAL

BASICS OF HYDROSTATIC SPINDLES THEORETICAL ANALYSIS: ANALYTICAL AND NUMERICAL APPROACHES

1. Hydrostatic bearings are a kind of contactless bearings. Rotating shaft and housing have no direct contact and they are separated by the thin layer of incompressible oil. Another example of contactless bearing is air static ones.
   Stiffness and load capacity of hydrostatic bearings is provided by external hydraulic power unit which supplies high pressure oil to the bearing recesses through the inlet restrictors. When external load is applied to the shaft, it will move from its initial position, and pressures in bearings recesses will be changed. Pressures differences between recesses will generate a bearing reaction force.
2. Hydrostatic spindles uniquely combine extremely high rotational accuracy, high stiffness and load capacity, ultra-high vibrations resistance (damping coefficient), very good thermal stability and they have virtually not restricted lifetime.
   However, the total cost of hydrostatic spindle system can be significantly higher as compared to the ball and roller bearings spindles. While the cost of spindle itself could be about the same as for ball bearings spindle, the peripheral support equipment is more expensive. It means that makes sense to use hydrostatic spindles only for such applications if other spindle types are not able to provide required level of performance. In other words, they have to be used for applications when a unique combination of their advantages is critical. Later we will discuss some examples.
3. The main goal of current tutorial is to provide listeners with a powerful tool to analyze hydrostatic bearings and to estimate their characteristics using analytical and numerical approaches.
   Probably you had noticed that have been used word “estimation” and not “calculation”. Definitely, the powerful software and modern computers allows calculate spindle characteristics with very high accuracy. However, this accuracy will be fictional and not real because of manufacturing tolerances and tolerances in other initial data. Let see some examples.
   - The most critical parameters for hydrostatic bearings are radial and axial gaps. The usual tolerances for these gaps are between 2-4 microns/diameter. It is definitely possible to reduce tolerances down to 1 micron or even to 0.5 micron. However, reducing tolerances will increase spindle manufacturing cost almost exponentially. Manufacturer will need much more precise measuring equipment, very high precision grinding machines and much more tough requirements to the temperature control stability in grinding, inspection and assembling departments;

- Oil viscosity is usually defined by the oil manufacturer with allowed deviations from the nominal value about 10%;
- The same is true for so critical parameter as elasticity modulus and even more – for the Poisson ratio.

The next inherent problem of numerical analysis is a number of bearing initial parameters. For the hydrostatic journal bearing this number can easily exceed 10-15 independent values. Without deep understanding and without intuitive feeling of spindle behavior, analyzing of hundreds of charts will be a real challenge.

Some important spindle characteristics as maximal load capacity or spindle dynamics, as transfer functions, resonances and stability, can be analyzed only using a numerical approach. However, it will be much clearer to interpret these numerical results after preliminary analytical investigation was made, and after researcher will get an intuitive feeling about the physical processes in the spindle bearings. As a little bit later could be seen, almost all main bearing characteristics can be analytically derived in the linear approach and it will be shown, that these characteristics are pretty accurate if spindle shaft moves from its initial position not more than a half of radial gap.

4. It is amazing, but the quite complete theoretical analysis will require only two simple basic formulas from hydrodynamics: Couette formula for incompressible oil flow through the flat gap (see Fig. 1)

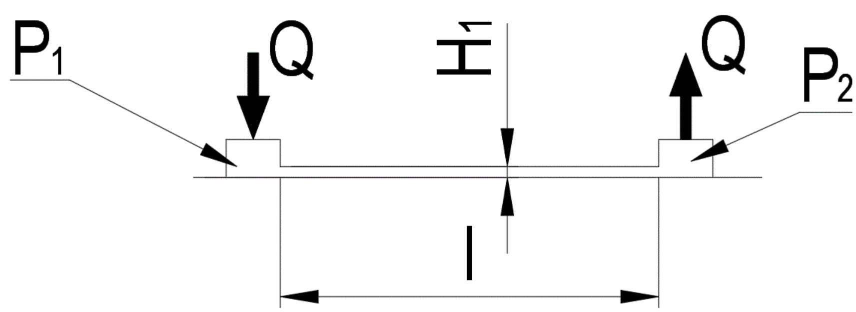


Fig. 1

Flow through the flat gap

$$Q = \frac{H_1^3 \, L}{12 \, \mu \, l} \, (P_1 - P_2) \tag{1}$$

And for friction power N generated by the shaft rotation in the small annular gap between shaft and housing (see Fig. 2)

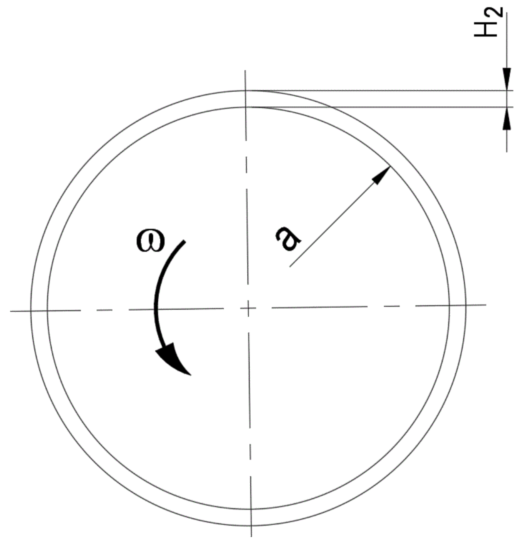


Fig. 2

$$N = \frac{\mu\,\omega^2\,a^2}{H_2}\,S = \frac{\mu\,\omega^2\,D^2}{4\,H_2}\,S \qquad (2)$$

Where H_1- is a size of the gap; l - width of the gap; L - is a length of the gap in direction perpendicular to the drawing; μ - is a coefficient of dynamic viscosity; Q - is a flow rate; P_1- oil pressure at beginning of the gap; P_2- oil pressure at end of the gap; ω - shaft rotational frequency; H_2- Is a radial gap; R - is an average radius of the small gap;
$D = 2a$; S - is the surface of the gap. Coefficient between flow and pressure difference we will designate as $\frac{1}{R}$ analogically as it common in electricity in the Ohm law. Flow is analogue to the current and pressures difference is analogue to the voltage.
Has to be noted that the basic equation for the oil lubrication (so named Reynolds equation) can be easily derived from the formula (1) for Couette flow in both Cartesian and in polar coordinates.

5. We will start from the simplest type of hydrostatic bearing – from the thrust bearing (see Fig. 3). It is important because only in this case it is possible to get simple analytical expressions for stiffness in both linear and non-linear approaches. However, the all procedures during analyzing this thrust bearing will be exactly the same for any other type of hydrostatic bearing.

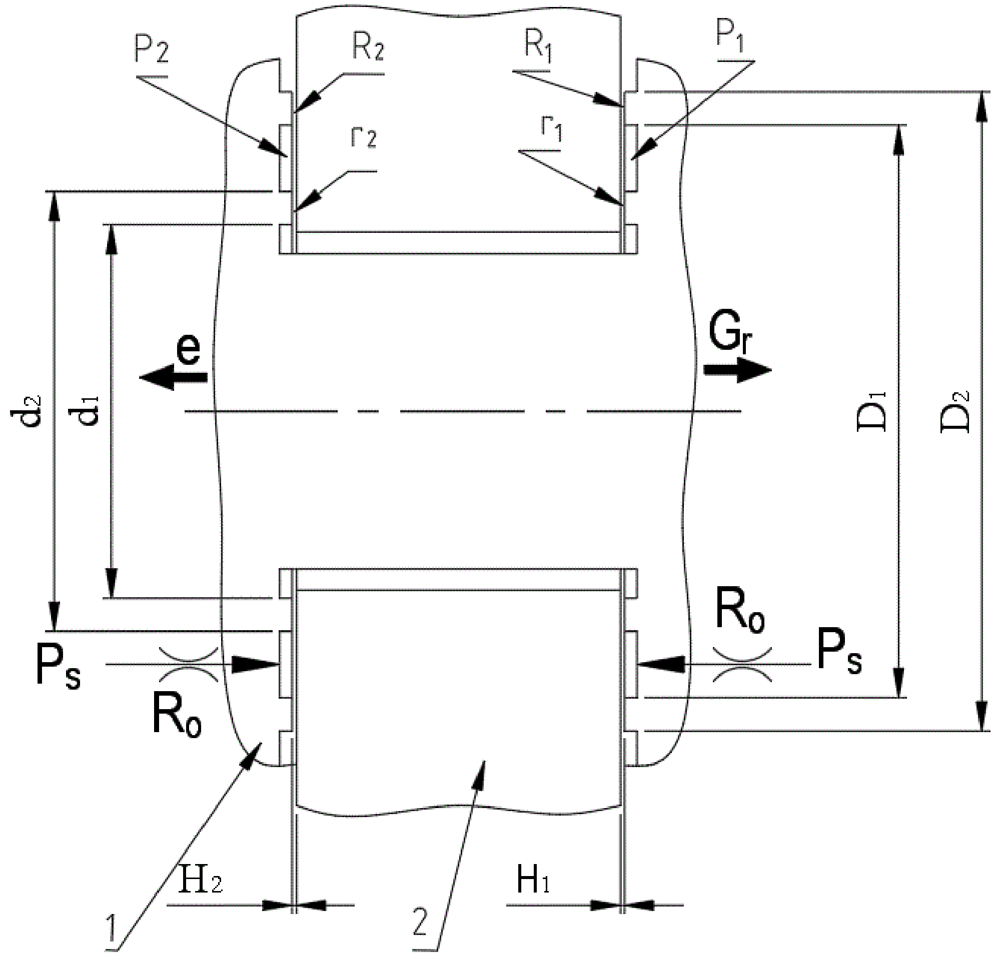


Fig. 3

Thrust Hydrostatic Bearing

On the Picture 3 we can see a typical example of the thrust hydrostatic bearing. There are two equal size chambers. Oil with the constant inlet pressure P_s is supplied to these chambers through the inlet restrictors R_0. Oil from the thrust bearing chambers is leaving to the oil return chambers through the annular gaps $h_1$and h_2. Oil pressure in the return line chambers is equal to atmospheric pressure and will be taken 0.

First of all, has to be clarified why we need inlet restrictors. Without inlet restrictors pressure in the thrust bearing chambers will be maximal, but the total reaction force applied to the shaft will be zero.

Our goal is not to get a maximal pressure in the thrust bearing chambers, but to get a maximal pressure difference between chambers at the given shaft axial displacement. In this case, the bearing stiffness will be maximal as well.

There are two types of the constant inlet restrictors: with laminar flow as small gaps or capillary with small internal opening diameter, and with turbulent flow as orifices. For the laminar flow restrictors, the flow through restrictor is proportional to the pressures difference, while in case of orifices, the flow is proportional to the square root from the pressures difference. For the current analysis will be used example with linear restrictors, but later will be shown how to analyze thrust bearing with orifice type restrictors.

To calculate stiffness and load capacity we have to find pressures in the thrust bearing chambers as functions of the axial displacement$e$.

For every hydrostatic bearing the initial equations for the pressures in the bearing chambers are flow balance equations: the flow coming into the chamber has to be equal to the flow leaving the chamber.

In our case we have two independent flow balance equations: for the left chamber with pressure P_1 and for the right chamber with the pressure P_2

$$\frac{P_S-P_1}{R_0}=\frac{P_1}{r_1}+\frac{P_1}{R_1} \qquad\qquad \frac{P_S-P_2}{R_0}=\frac{P_2}{r_2}+\frac{P_2}{R_2} \tag{3}$$

Formulas for outlet restrictors $r_1, r_2, R_1$and R_2 if shaft moved by external load on the distance e from the central position are as follow

$$\frac{1}{r_1}=\frac{\pi\,(h_0-e)^3}{6\,\mu\ln\frac{d_2}{d_1}} \qquad\qquad \frac{1}{R_1}=\frac{\pi\,(h_0-e)^3}{6\,\mu\ln\frac{D_2}{D_1}} \tag{4}$$

$$\frac{1}{r_2}=\frac{\pi\,(h_0+e)^3}{6\,\mu\ln\frac{d_2}{d_1}} \qquad\qquad \frac{1}{R_2}=\frac{\pi\,(h_0+e)^3}{6\,\mu\ln\frac{D_2}{D_1}} \tag{5}$$

Formulas (4) and (5) are a simple modification of the Couette formula (1) for the case of annular gaps in the thrust bearing. Later, it will be shown how these formulas can be derived, but right now I don't want to interrupt an explanation.

The h_0 in formulas (4) and (5) is an initial gap in the thrust bearing when external load is not applied and shaft axial movement from its central position is 0.

The next step will be to change variables and to use a relative non dimensional eccentricity ε instead of absolute eccentricity e

$$\varepsilon=\frac{e}{h_0} \tag{6}$$

Formulas (4) can be rewritten

$$\frac{1}{r_1}=\frac{\pi\, h_0^3\,(1-\varepsilon)^3}{6\,\mu\,\ln\frac{d_2}{d_1}}\equiv\frac{1}{r_{10}}(1-\varepsilon)^3 \qquad \frac{1}{R_1}=\frac{\pi\, h_0^3\,(1-\varepsilon)^3}{6\,\mu\,\ln\frac{D_2}{D_1}}\equiv\frac{1}{R_{10}}(1-\varepsilon)^3 \qquad (7)$$

$$\frac{1}{r_2}=\frac{\pi\, h_0^3\,(1+\varepsilon)^3}{6\,\mu\,\ln\frac{d_2}{d_1}}\equiv\frac{1}{r_{10}}(1+\varepsilon)^3 \qquad \frac{1}{R_2}=\frac{\pi\, h_0^3\,(1+\varepsilon)^3}{6\,\mu\,\ln\frac{D_2}{D_1}}\equiv\frac{1}{R_{10}}(1+\varepsilon)^3 \qquad (8)$$

Where r_{10} and $R_{10}$are values for $r_1, r_2$and for $R_1, R_2$when shaft is in the central position.

By substituting (7) and (8) into (3) one can get the following equations for pressures in the chambers $P_1$and P_2

$$P_S\,\frac{1}{R_0}=P_1\left[\frac{1}{R_0}+(1-\varepsilon)^3\,\left(\frac{1}{r_{10}}+\frac{1}{R_{10}}\right)\right] \qquad (9)$$

$$P_S\,\frac{1}{R_0}=P_2\left[\frac{1}{R_0}+(1+\varepsilon)^3\,\left(\frac{1}{r_{10}}+\frac{1}{R_{10}}\right)\right] \qquad (10)$$

Now will be introduced a new non-dimensional variable λ that defined as follows

$$\frac{1}{r_{10}}+\frac{1}{R_{10}}=\lambda\,\frac{1}{R_0} \qquad (11)$$

By substituting (11) in (9) and (10), equations for pressures in the chambers will be transformed

$$P_1[1+(1-\varepsilon)^3\,\lambda]=P_S \qquad P_2[1+(1+\varepsilon)^3\,\lambda]=P_S \qquad (12)$$

As it can be seen from equations (12), the axial bearing behavior now is described by only one parameter λ.

For the bearing reaction force G_R applied to the shaft we have

$$G_R(\varepsilon)=[P_1(\varepsilon)-P_2(\varepsilon)]\,S_0 \qquad (13)$$

Where S_0 is surface of the thrust bearing chamber.

Finding pressures from formulas (12) and substituting them into equation (13), one can get

$$G_R(\varepsilon)=2\,\lambda\,P_S\,S_0\,\frac{3\varepsilon+\varepsilon^2}{1+2\,\lambda\,(1+3\,\varepsilon^2)+\lambda^2(1-\varepsilon^2)^3} \qquad (14)$$

For the reaction force $G_R(\varepsilon)$ when shaft is close to the central position, the linear interpolation can be used and all members with ε power 2 and higher can be neglected. In this case formula (14) will be significantly simplified

$$G_R(\varepsilon) = 6\,\varepsilon\, P_S S_0 \frac{\lambda}{(1+\lambda)^2} \tag{15}$$

Axial stiffness C_{AX} in the central position will be described by the following formula

$$C_{AX}(e=0) = \frac{dG_R}{de} = \frac{1}{h_0}\frac{dG_R}{d\varepsilon} = \frac{6\,P_S\,S_0}{h_0}\;\frac{\lambda}{(1+\lambda)^2} \tag{16}$$

It easy can be verified that stiffness will be maximal when parameter $\lambda = 1$

So, there is a final formula for the maximal axial stiffness in the central position

$$C_{AX} = \frac{3\,P_S\,S_0}{2\,h_0} \tag{17}$$

The found optimal value 1 for parameter λ means that stiffness will be maximal if inlet restrictor to the chamber is equal to the total outlet restrictor from the chamber as can be seen from the formula (11). The pressures in the chambers at the shaft central position will be equal to half of supply pressure that can be seen from formulas (12). By measuring pressure in thrust bearing chamber can be verified if spindles have been designed and manufactured correctly.

Formulas (11) have to be used to calculate an optimal value of the inlet restrictor R_0 in dependence of outlet restrictors values r_{10} and $R_{10.}$

Now we are ready to calculate damping coefficient and to find shaft motion equations.

Let us take that shaft moved to the right in axial direction on the distance e and has a speed $\frac{de}{dt}$ in the same direction. In this case the flow balance equations (3) has to be modified.

For the right chamber will be an additional inlet flow Q caused by the shaft speed, and for the left chamber will be an additional outlet flow of the same value Q. The modified flow balance equations will be as follows

$$\frac{P_S-P_1}{R_0} + S_0\frac{de}{dt} = \frac{P_1}{r_1} + \frac{P_1}{R_1} \qquad\qquad \frac{P_S-P_2}{R_0} = S_0\frac{de}{dt} + \frac{P_2}{r_2} + \frac{P_2}{R_2} \tag{18}$$

Using the same variables as were used in stiffness analysis, can be found new expressions for pressures P_1 and P_2

$$P_1 = \frac{P_S}{1+\lambda(1-\varepsilon)^3} + \frac{R_0\,S_0\,h_0}{1+\lambda(1-\varepsilon)^3}\;\frac{d\varepsilon}{dt} \qquad P_2 = \frac{P_S}{1+\lambda(1+\varepsilon)^3} - \frac{R_0\,S_0\,h_0}{1+\lambda(1+\varepsilon)^3}\;\frac{d\varepsilon}{dt} \tag{19}$$

By substituting $P_1$and P_2 into the bearing reaction force equation (13) one will get

$$G_R\left(\varepsilon,\frac{d\varepsilon}{dt}\right) = \lambda\, P_S\, S_0\ \frac{(1+\varepsilon)^3+(1-\varepsilon)^3}{(1+\lambda(1+\varepsilon)^2)(1+\lambda(1-\varepsilon)^2)} +$$

$$+\ \ R_0\, S_0^2\, h_0\ \frac{2+\lambda(1+\varepsilon)^3+\lambda(1-\varepsilon)^3}{(1+\lambda(1+\varepsilon)^2)(1+\lambda(1-\varepsilon)^2)}\ \frac{d\varepsilon}{dt} \tag{20}$$

The first member in the right side of equation (20) describes static force generated by the shaft displacement from the central position, while the second member describes damping force that is generated by the shaft speed.

In the linear approach for variables ε and $\frac{d\varepsilon}{dt}$, and taking that parameter $\lambda = 1$ (to get maximal stiffness), equation for reaction force (20) could be significantly simplified

$$G_R\left(\varepsilon,\frac{d\varepsilon}{dt}\right) = \frac{3}{2}\ P_S\, S_0\, \varepsilon + R_0\, S_0^2\, h_0\, \frac{d\varepsilon}{dt} \tag{21}$$

Or, going back to the dimensional variable e (displacement), the equation (21) can be rewritten

$$G_R\left(e,\frac{de}{dt}\right) = \frac{3\, P_S\, S_0}{2\, h_0}\, e\ + R_0\, S_0^2\ \frac{de}{dt} \tag{22}$$

The coefficient at e is a static stiffness C; coefficient at $\frac{de}{dt}$ is a damping coefficient K which describes energy dissipation in the system

$$C = \frac{3\, P_S\, S_0}{2\, h_0} \qquad\qquad K = R_0\, S_0^2 \tag{23}$$

The shaft motion equation now can be written in the standard form

$$M\frac{d^2e}{dt^2} + K\ \frac{de}{dt} + Ce = F(t) \tag{24}$$

Where M– mass of the shaft; $F(t)$ - external alternate load applied to the shaft in the axial direction.

The very simple way to estimated accuracy of the linear approach is to use symmetry of the thrust bearing. Because of symmetry, reaction force at relative eccentricity ε has to be exactly the same as at relative eccentricity $-\varepsilon$, but applied in opposite direction

$$G_R(\varepsilon) \equiv -G_R(-\varepsilon) \qquad (25)$$

By expanding these functions into the Taylor series, one will get

$$G_R(\varepsilon) = G_R(0) + a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + \cdots = a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + \ldots$$

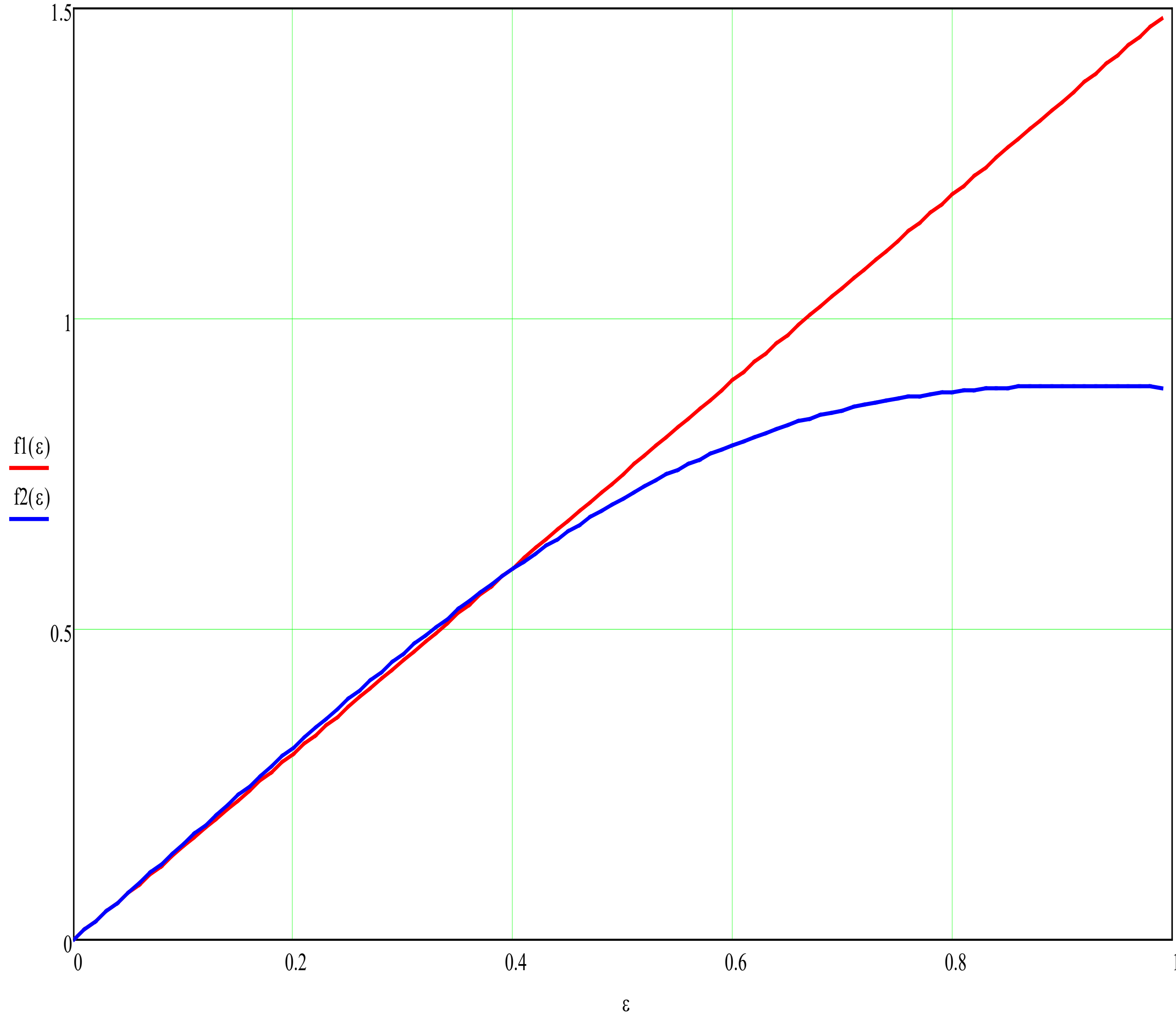
$$G_R(-\varepsilon) = G_R(0) - a_1\varepsilon + a_2\varepsilon^2 - a_3\varepsilon^3 + \cdots = -a_1\varepsilon + a_2\varepsilon^2 - a_3\varepsilon^3 + \ldots \qquad (26)$$

It was assumed that $G_R()) = 0$.

After substituting (26) into (25) we will get the following identity formula

$$a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 \ldots \equiv a_1\varepsilon - a_2\varepsilon^2 + a_3\varepsilon^3 \ldots \qquad (27)$$

To have left and right sides in (27) identical, the coefficient a_2 must be 0. Therefore, the linear approach error will start with ε^3 because member with ε^2 is disappeared. It can be seen as well on the linear and non-linear approaches charts that can be easily calculated from formulas (14) and (15)



AXIAL LOAD CAPACITY VS RELATIVE ECCENTRICITY

RED LINE – LINEAR APPRCOACH; BLUE LINE – NON-LINEAR APPROACH

The next topic will be stiffness analysis for the thrust hydrostatic bearing with orifice inlet restrictors.

As it was told earlier, the flow rate through the orifice is proportional to a square root from the pressure drop in the orifice

$$Q = \beta \sqrt{\Delta P} \tag{28}$$

The flow balance equations will be similar to formulas (3)

$$\beta \sqrt{P_S - P_1} = \frac{P_1}{r_1} + \frac{P_1}{R_1} \qquad\qquad \beta \sqrt{P_S - P_2} = \frac{P_2}{r_2} + \frac{P_2}{R_2} \tag{29}$$

The most evident way is to convert equations (29) into two quadratic equations for pressures P_1 and P_2. However, using this way will require long calculations. Besides that, conversion to the quadratic equations will create additional roots which are not corresponding to initial equations (29).

To perform stiffness analysis and parameters optimization we will start from the formulas (13) and (16). Axial stiffness at the shaft central position can be written as follows

$$(C_{AX})_{e=0} = \left(\frac{dG_R}{de}\right)_{e=0} = \frac{1}{h_0}\left(\frac{dG_R}{d\varepsilon}\right)_{\varepsilon=0} = \frac{S_0}{h_0}\left(\left(\frac{dP_1}{d\varepsilon}\right)_{\varepsilon=0} - \left(\frac{dP_2}{d\varepsilon}\right)_{\varepsilon=0}\right) \tag{30}$$

Instead to find unknown pressures from equations (29) and to take derivatives, the equations (29) can be converted to equations directly for pressures derivatives. Taken derivations from equations (29) and by turning the eccentricity $\varepsilon = 0$, one will get

$$-\frac{1}{2}\,\beta\,(P_S - P_0)^{-\frac{1}{2}}\left(\frac{dP_1}{d\varepsilon}\right)_{\varepsilon=0} =$$
$$= \left(\frac{1}{r_{10}} + \frac{1}{R_{10}}\right)\left(\frac{dP_1}{d\varepsilon}\right)_{\varepsilon=0} + P_0\left[\left(\frac{d}{d\varepsilon}\frac{1}{r_1}\right)_{\varepsilon=0} + \left(\frac{d}{d\varepsilon}\frac{1}{R_1}\right)_{\varepsilon=0}\right] \tag{31}$$

$$-\frac{1}{2}\,\beta\,(P_S-P_0)^{-\frac{1}{2}}\left(\frac{dP_2}{d\varepsilon}\right)_{\varepsilon=0}=$$

$$=\left(\frac{1}{r_{10}}+\frac{1}{R_{10}}\right)\left(\frac{dP_2}{d\varepsilon}\right)_{\varepsilon=0}+\;P_0\left[\left(\frac{d}{d\varepsilon}\frac{1}{r_2}\right)_{\varepsilon=0}+\left(\frac{d}{d\varepsilon}\frac{1}{R_2}\right)_{\varepsilon=0}\right] \tag{32}$$

Equations (31) and (32) can be converted to the following equations for pressures derivatives at central position ($\varepsilon=0$)

$$\left(\frac{dP_1}{d\varepsilon}\right)_{\varepsilon=0}\left[\frac{1}{r_{10}}+\frac{1}{R_{10}}+\frac{1}{2}\beta(P_S-P_0)^{-\frac{1}{2}}\right]=3\,P_0\left(\frac{1}{r_{10}}+\frac{1}{R_{10}}\right) \tag{33}$$

$$\left(\frac{dP_2}{d\varepsilon}\right)_{\varepsilon=0}\left[\frac{1}{r_{10}}+\frac{1}{R_{10}}+\frac{1}{2}\beta(P_S-P_0)^{-\frac{1}{2}}\right]=-3\,P_0\left(\frac{1}{r_{10}}+\frac{1}{R_{10}}\right) \tag{34}$$

Now will be introduced a new parameter α

$$\alpha=\frac{P_0}{P_S} \tag{35}$$

Parameter β can be expressed through the parameter α using a flow balance equation at the shaft central position (one of equations (29) at $\varepsilon=0$)

$$\beta=\frac{\alpha\sqrt{P_S}}{\sqrt{1-\alpha}}\left(\frac{1}{r_{10}}+\frac{1}{R_{10}}\right) \tag{36}$$

By substituting (35) and (36) into equations (33) and (34), the formulas for pressures derivatives can be received

$$\left(\frac{dP_1}{d\varepsilon}\right)_{\varepsilon=0}=6\,P_S\,\frac{\alpha\,(1-\alpha)}{2-\alpha}\qquad\left(\frac{dP_2}{d\varepsilon}\right)_{\varepsilon=0}=-6\,P_S\,\frac{\alpha\,(1-\alpha)}{2-\alpha} \tag{37}$$

By substituting (37) into (30) one will get the final simple expression for the axial stiffness at the shaft central position

$$(C_{AX})_{\varepsilon=0}=\frac{12\,P_S\,S_0}{h_0}\,\frac{\alpha\,(1-\alpha)}{2-\alpha} \tag{38}$$

It can be easy verified that maximal stiffness will be reached if parameter $\alpha=2-\sqrt{2}\approx 0.59$.

Orifice's parameter β can be expressed using equation (36) through parameter α, supply pressure P_S and outlet restrictors values r_{10} and R_{10}.

Our considerations about relation between bearing symmetry and accuracy of the linear approach can be evidently applied as well for the case with orifice type inlet restrictors.

The last topic related to thrust bearing analysis will be the friction power calculation. For that purpose, will be used a slightly modified formula (2). Formula (2) describes friction power in the radial gap with constant gap's radius, while in the thrust bearing the radius of the gap is changing from value a to value b. On the Fig. 4 is shown a gap located on the thrust surface with radius changing from $r = a$ to $r = b$. The surface element dS between radius r and radius $r + dr$ is described by the following formula

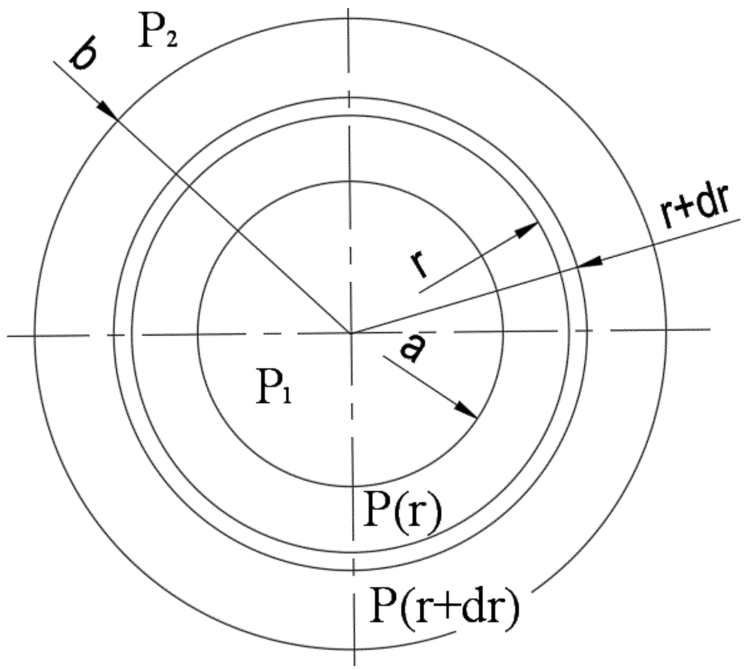


Fig. 4

Axial gap in the thrust bearing

$$dS = \pi\,(r + dr)^2 - \pi r^2 = 2\,\pi\,r\,dr \qquad (39)$$

Using formula (2) one can get the expression for the power element dN

$$dN = \frac{\mu\,\omega^2\,r^2}{H} dS = \frac{2\,\pi\,\mu\,\omega^2}{H}\,r^3\,dr \qquad (40)$$

For the total friction power in the gap we have

$$N = \frac{2\,\pi\,\mu\,\omega^2}{H}\,\int_a^b r^3\,dr = \frac{\pi\,\mu\,\omega^2}{2\,H}\,(b^4 - a^4) \qquad (41)$$

In our case, according to the Fig. 2

$$a = 0.5\, d_1 \quad \text{or} \quad a = 0.5\, D_1 \qquad\qquad b = 0.5\, d_2 \quad \text{or} \quad b = 0.5\, D_2 \tag{42}$$

Now, to complete thrust bearing analysis will be shown how Couette formula for the flat gap (see Picture 1) can be modified for the gaps in the thrust bearing.

Let us consider a flat annular gap with internal radius a and with external radius b (see Picture 4). Pressure at the radius a will be designated as P_1 and pressure at the radius b will be designated as P_2. Now it will be considered two closely located cross-sections with radius r and with radius $r + dr$. Pressure at these cross-sections will be $P(r)$ and $P(r) + dP(r)$. The length of the gap will be evidently $2\pi r$. Therefore, in formula (1) has to be made changes

$$\frac{P_1 - P_2}{l} \;\rightarrow\; -\frac{dP}{dr} \qquad\qquad L \;\rightarrow\; 2\,\pi\, r \tag{43}$$

Formula for the flow now can be written as follows

$$Q = -\frac{\pi\, H_1^3}{6\,\mu}\left(r\,\frac{dP}{dr}\right) \tag{44}$$

Because the oil is considered incompressible, the radial flow in every cross-section has to be constant, that means that flow derivative taken by the variable r has to be zero

$$\frac{dQ}{dr} = 0 \quad \rightarrow \quad \frac{d}{dr}\left(r\,\frac{dP}{dr}\right) = 0 \tag{45}$$

Solving this very simple equation with boundary conditions $P(a) = P_1$ and $P(b) = P_2$ one will get the following formulas for the pressure distribution $P(r)$ and for the flow Q

$$P(r) = \frac{P_1 - P_2}{\ln\frac{a}{b}}\,\ln r + \frac{P_2\,\ln a - P_1\,\ln b}{\ln\frac{a}{b}} \qquad\qquad r\,\frac{dP}{dr} = \frac{P_1 - P_2}{\ln\frac{a}{b}} \tag{46}$$

$$Q = \frac{\pi\, H_1^3\,(P_1 - P_2)}{6\,\mu\,\ln\frac{b}{a}} \tag{47}$$

Next topic will be related to journal bearing analysis.

On the Fig. 5 is shown a longitudinal view of the typical hydrostatic spindle. Spindle contains front and rear journal bearings 3 and 7, and thrust bearing 5. Shaft 1 is mounted into the housing 2. Both front journal 3 and rear journal 7 have radial stiffness, but their tilting stiffness is zero. To provide tilting stiffness to the spindle, two journal bearings have to be used.

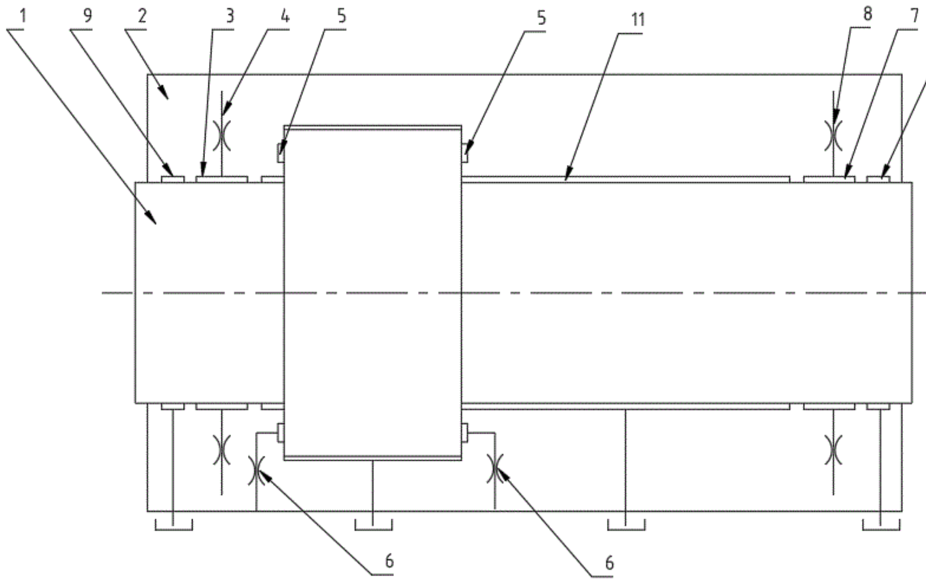


Figure 5

Hydrostatic Spindle

Through the inlet restrictors 4 and 8 high pressure oil from external hydraulic power unit is supplied to journal bearings recesses, and through the inlet restrictors 6, high pressure oil is supplied to the thrust bearing chambers. From journal bearings recesses and from thrust bearing chambers oil is leaving to the annular return line chambers 9,10,11, 12 and 13 that are directly connected to the hydraulic unit return line.

On the Fig. 6 is shown cross-sectional view of journal bearing. Number of recesses is chosen 4, but it can be 5, 6, 7 etc. Journal bearing with two recesses will have radial stiffness only in one direction. The three recesses journal bearing will have radial stiffness value which is very sensitive to the external load direction.

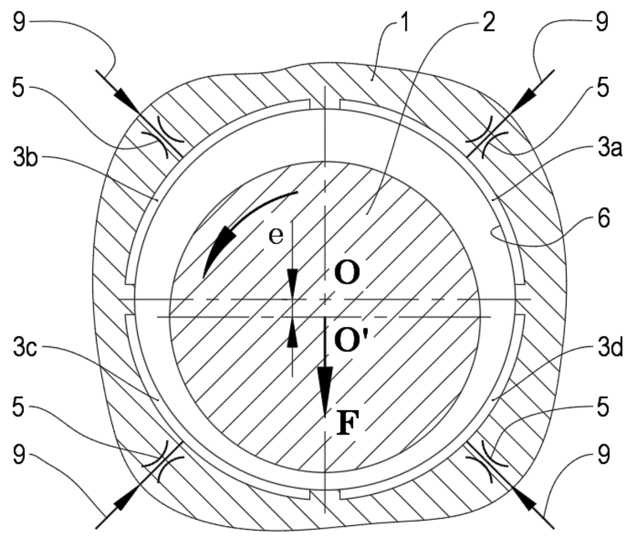


Figure 6

Cross-sectional view of Hydrostatic Journal Bearing

Let us consider an external load Fapplied to the shaft in vertical direction. As a result, shaft will move on certain distance e from its central position. Outlet gaps in two bottom recesses will be reduced, while an outlet gaps in two upper recesses will be increased. So, pressures in the bottom recesses will grow, while pressures in the upper recesses will drop. Pressures difference between bottom recesses and upper recesses will generate a bearing reaction force directed against the external load F.

As it was in the thrust bearing analysis, the initial equations for pressures in recesses will be flow balance equations. But there are two main issues that make these flow balance equations significantly more sophisticated as compared to the thrust bearing.

1. When shaft is moved in radial direction from its central position, the radial gap will be not uniform and will depend of angular coordinate.

2. Besides oil flows from recesses to the return line chambers, will be additional flows between neighboring recesses and, therefore, every equation will contain at least three unknowns: pressure in the recess and pressures in the two neighboring recesses.

To write flow balance equations have to be calculated outlet restrictors values for the axial flows from recesses to the return line chambers and for tangential flows between neighboring recesses. To calculate these flows will be used formula for the radial gap $h(\phi)$ as function of the angular coordinateϕ. If shaft is moved on the distance e from the central position in the negative direction if the y -axis, the gap function will be described by the following formula (see Fig. 7)

$$h(\varphi) = h_0 - e\ \cos(\varphi - \psi) = h_0(1 - \varepsilon\ \cos(\varphi - \psi)) \tag{48}$$

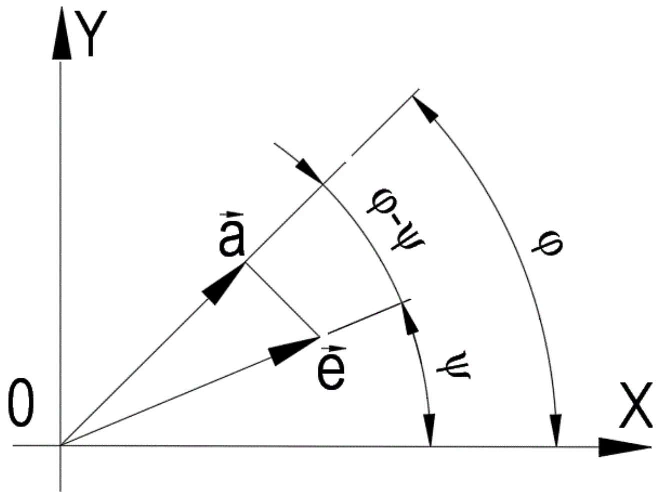


Figure 7

The axial flow $Q(i)$ form recess number i to the return line chamber can be calculated as follows using Couette formula

$$dQ(i) = P_i \frac{h_0^3\,(1+\varepsilon\ \sin\varphi)^3}{12\,\mu\ L}\ R\ (d\varphi)\ \rightarrow$$

$$\rightarrow Q(i) = P_i \frac{h_0^3\,R}{12\,\mu\,L} \int_{\alpha_i}^{\beta_i} (1 + \varepsilon\ \sin\varphi)^3\ d\varphi \tag{49}$$

Where h_0 is a radial gap at the shaft central position;ε is a relative eccentricity; R is a shaft radius; μ is an oil dynamic viscosity coefficient; L is a width of the gap between radial recess and the return line chamber; $\alpha_i$and $\beta_i$are angular coordinates of the recess with number i.

If number of recesses is n and initial angle of the first recess is γ, the angles $\alpha_i$and $\beta_i$can be written as follows (see Fig. 8)

$$\alpha_i = \gamma + \frac{2\pi}{n}(i-1) \qquad\qquad \beta_i = \gamma + \frac{2\pi}{n} i \tag{50}$$

Figure 8

In the linear approach over variable ε one can get

$$Q(i) = P_i \frac{h_0^3\, R}{12\,\mu\;\, L}\left(\frac{2\pi}{n} + 6\varepsilon\;\sin\frac{\pi}{n}\;\sin\left(\gamma + \frac{\pi}{n}(2i-1)\right)\right) \qquad (51)$$

The gap between recesses with number i and number $i-1$ has an angular coordinate $\alpha_{i,}$and the gap between recesses with number i and number $i+1$ has an angular coordinate β_i. Therefore flows $Q(i-1,i)$ and $Q(i,i+1)$ from recess number $i-1$ to recess number i and from recess number i to recess number $i+1$ will be described by the formulas

$$Q(i-1,i) = (P_i - P_{i-1})\frac{h_0^3\, R}{12\,\mu\, L}(1 + 3\varepsilon\;\sin\alpha_i) \qquad (52)$$

$$Q(i,i+1) = (P_i - P_{i+1})\frac{h_0^3\, R}{12\,\mu\, L}(1 + 3\varepsilon\;\sin\beta_i) \qquad (53)$$

Formulas (51)-(53) can be rewritten as follows

$$Q(i) = P_i \frac{1}{r_{10}}\left(1 + \varepsilon\ \frac{3n}{\pi}\ \sin\frac{\pi}{n}\ \sin\left(\gamma + \frac{\pi}{n}(2i-1)\right)\right) \tag{54}$$

$$Q(i-1,i) = (P_i - P_{i-1})\frac{1}{r_{120}}(1 + 3\varepsilon\ \sin\alpha_i) \tag{55}$$

$$Q(i,i+1) = (P_i - P_{i+1})\frac{1}{r_{120}}(1 + 3\varepsilon\ \sin\beta_i) \tag{56}$$

Where r_{10} is a value for restrictor to the axial flow at zero eccentricity; r_{120}- value for restrictors to tangential flows between recesses at zero eccentricity

$$\frac{1}{r_{10}} = \frac{\pi\ D\ h_0^3}{12\ \mu\ n\ L} \qquad\qquad \frac{1}{r_{120}} = \frac{L_e\ h_0^3}{12\ \mu\ L_2} \tag{57}$$

Now we are ready to write flow balance equations for the recess with number i

$$\frac{P_S - P_i}{R_0} = 2P_i \frac{1}{r_{10}}\left(1 + \varepsilon\ \frac{3n}{\pi}\ \sin\frac{\pi}{n}\ \sin\left(\gamma + \frac{\pi}{n}(2i-1)\right)\right) +$$

$$+(P_i - P_{i-1})\frac{1}{r_{120}}(1 + 3\varepsilon\ \sin\alpha_i) + (P_i - P_{i+1})\frac{1}{r_{120}}(1 + 3\varepsilon\ \sin\beta_i) \tag{58}$$

$i = 1, 2, 3 \ldots n$

Here R_0- is an inlet restrictor's value.

The next step will be to calculate a reaction force through the pressures in the recesses.

Let us take a small segment with angle $d\phi$ in the recess number i with pressure P_i (Fig. 9).

Pressure P_i is constant inside the recess. For the force element dF applied to the shaft from pressure one will get

$$dF_i = P_i\ dS = P_i L_e R\ d\varphi \tag{59}$$

Here L_e is an effective length of the recess.

For the force dF_i projection on the y-axis dF_{iy} we have

$$dF_{iy} = -dF_i \sin\varphi = -P_i L_e R\ \sin\varphi\ d\varphi \tag{60}$$

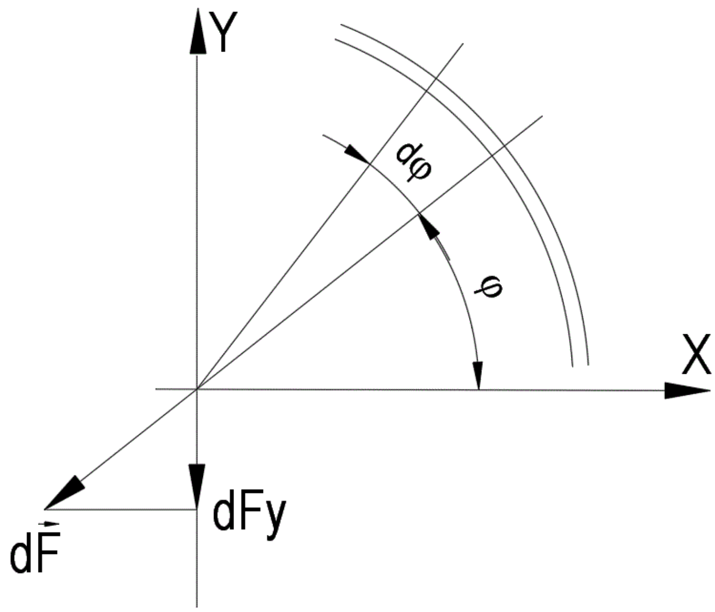


Fig. 9
Force Calculation

Therefore, the reaction force $F_y(i)$ applied to the shaft from the recess number i in direction of the y -axis will be as follows

$$F_y(i) = -P_i\, L_e R\, \int_{\alpha_i}^{\beta_i} \sin\varphi\, d\varphi = -2P_i\, L_e R\, \sin\frac{\pi}{n} \sin\left(\gamma + \frac{\pi}{n}(2i-1)\right)$$

(61)

Making a summation over the number$i$ from $i = 1$ to $i = n$, the total force applied to the shaft in y direction from all recesses will be

$$F_y = \sum_{i=1}^{n} F_{iy} = -2L_e R\, \sin\frac{\pi}{n} \sum_{i=1}^{n} P_i \sin\left(\gamma + \frac{\pi}{n}(2i-1)\right) \qquad (62)$$

$$C_y(e=0) = \frac{1}{h_0}\left(\frac{dF_y}{d\varepsilon}\right)_{\varepsilon=0} = \frac{2L_e R}{h_0} sin\frac{\pi}{n} \sum_1^n \left(\frac{dP_i}{d\varepsilon}\right)_{\varepsilon=0} sin\left(\gamma + \frac{\pi}{n}(2i-1)\right)$$

Has to be emphasized, that equations system (58) can be transformed to one linear equation for the stiffness and there is no need to solve equation system (54). It makes able to get simple and compact analytical expressions for bearing radial stiffness and for damping coefficient as well. However, the detail explanation will take much more time that we have in our disposal.

Only as a reference, the formulas for radial stiffness C and for damping coefficient K for the spindle shown on the Fig. 5, will be as follows

$$C = \frac{0.375\, n\, P_S L_e D}{h_0} \frac{1}{\frac{\pi}{n\left(sin\frac{\pi}{n}\right)^2} + \frac{L_e\, L_1}{L_3\, D}} \qquad K = \frac{1.5\, \mu\, n\, L_e^2\, L_1 D}{h_0^3} \frac{1}{\frac{\pi}{n\left(sin\frac{\pi}{n}\right)^2} + \frac{L_e\, L_1}{L_3\, D}} \qquad (63)$$

Here n is number of recesses; P_S- supply pressure; L_e- effective length of the recess; D - bearing diameter; h_0- radial gap at the shaft central position; L_1- width of the gap between recess and return line chamber; L_3- width of the gap between recesses.

It has to be noted that in linear approach radial stiffness and damping coefficient are not depending upon direction of the shaft displacement.

There are two kinds of hydrostatic journal bearings that stiffness can be easily calculated without to resolve system of equations. It is journal bearings with 4 recesses and with 6 recesses (see Fig. 10a and Fig. 10b). For both bearings can be chosen such force direction that will be recesses in which pressure will remain constant if shaft displacements are small as compared to the radial gap. Usually they named as neutral recesses. For the 6 recesses version, pressures in two upper recesses will be equal as well as pressures in two bottom recesses. Pressure in the neutral recesses can be calculated from the flow balance equation for the shaft central position.

As example we will analyze 4 recesses bearing. The flow balance equations for the upper recess and for the bottom recess will be

$$\frac{P_S - P_1}{R_0} = 2\frac{P_1}{r_1} + 2\frac{P_1 - P_0}{r_{12}} \quad \rightarrow \quad \frac{P_S}{R_0} = P_1\left(\frac{1}{R_0} + \frac{2}{r_1} + \frac{2}{r_{12}}\right) - 2\frac{P_0}{r_{12}} \qquad (64)$$

$$\frac{P_S - P_2}{R_0} = 2\frac{P_2}{r_2} + 2\frac{P_2 - P_0}{r_{23}} \quad \rightarrow \quad \frac{P_S}{R_0} = P_2\left(\frac{1}{R_0} + \frac{2}{r_2} + \frac{2}{r_{23}}\right) - 2\frac{P_0}{r_{23}} \qquad (65)$$

Here r_{12} is a hydraulic resistance between upper recess and neutral recesses; r_{23} is a hydraulic resistance between bottom recesses and the neutral recesses.

Taking into account that $n = 4$, from formulas (54), (55), (56) and (62) we will get following expressions for hydraulic resistances r_1, r_2, r_{12} and r_{23} , and for reaction force $F_y(\varepsilon)$

$$\frac{1}{r_1} = \frac{1}{r_{10}}\left(1 + \varepsilon\, \frac{6\sqrt{2}}{\pi}\right) \qquad \frac{1}{r_2} = \frac{1}{r_{10}}\left(1 - \varepsilon\, \frac{6\sqrt{2}}{\pi}\right) \qquad (66)$$

$$\frac{1}{r_{12}} = \frac{1}{r_{120}}\left(1 + \varepsilon\, \frac{3\sqrt{2}}{2}\right) \qquad \frac{1}{r_{23}} = \frac{1}{r_{120}}\left(1 - \varepsilon\, \frac{3\sqrt{2}}{2}\right) \qquad (67)$$

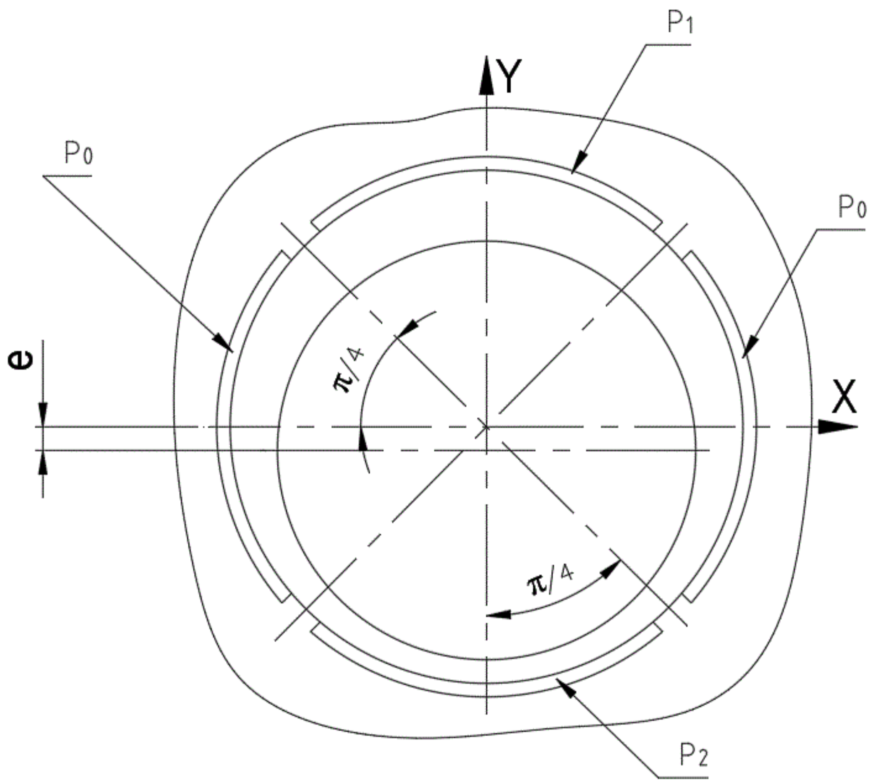


Fig. 10a
Journal Bearing with 4 Recesses

$$F_y(\varepsilon) = L_e\, D\, \frac{\sqrt{2}}{2}\left(P_2(\varepsilon) - P_1(\varepsilon)\right) \tag{68}$$

Therefore, we are getting for stiffness at the shaft central position

$$C(e=0) = \left(\frac{dF_y}{de}\right)_{e=0} = \frac{1}{h_0}\left(\frac{dF_y}{d\varepsilon}\right)_{\varepsilon=0} = \frac{\sqrt{2}\, L_e\, D}{2\, h_0}\left[\left(\frac{dP_2}{d\varepsilon}\right)_{\varepsilon=0} - \left(\frac{dP_1}{d\varepsilon}\right)_{\varepsilon=0}\right] \tag{69}$$

To find derivatives in (69) we will take derivatives from equations (64) and (65) and after, will turn ε to zero. We will get the following equations

$$\left(\frac{dP_1}{d\varepsilon}\right)_{\varepsilon=0}\left(\frac{1}{R_0} + 2\frac{1}{r_{10}} + 2\frac{1}{r_{120}}\right) = -\frac{12\sqrt{2}\, P_0}{\pi\, r_{10}} \tag{70}$$

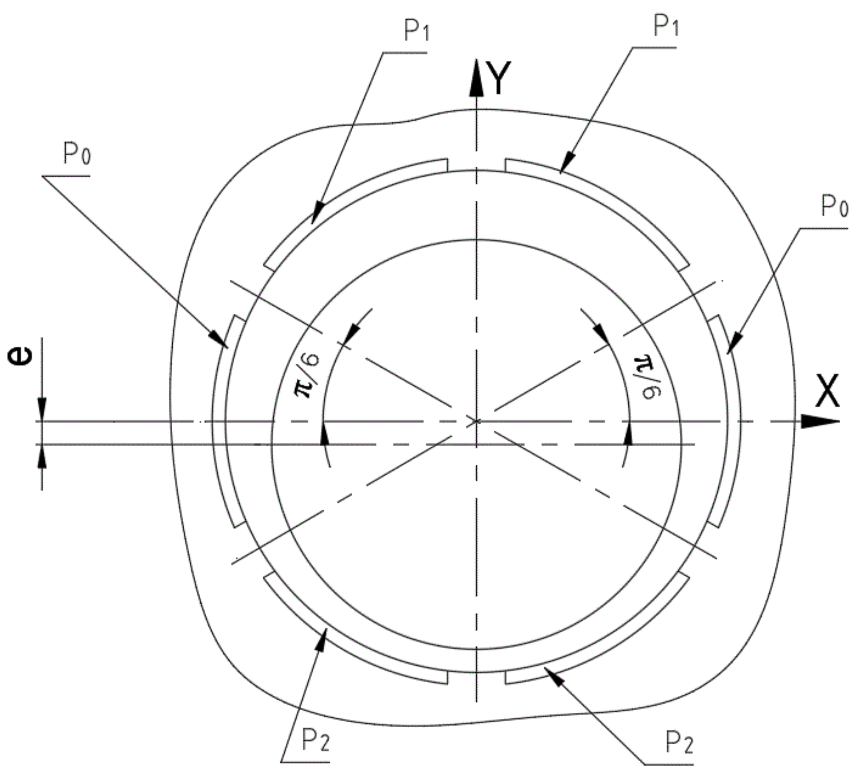


Fig. 10b
Journal Bearing with 6 Recesses

$$\left(\frac{dP_2}{d\varepsilon}\right)_{\varepsilon=0}\left(\frac{1}{R_0}+2\frac{1}{r_{10}}+2\frac{1}{r_{120}}\right)=\frac{12\sqrt{2}\,P_0}{\pi\, r_{10}} \tag{71}$$

By substituting (70) and (71) into (69) we are getting the formula for the stiffness expressed through the pressure in the recesses at central position and through hydraulic resistances taken as well at the shaft central position

$$C(e=0)=\frac{24\,L_e\,D\,P_0}{\pi\,h_0}\,\frac{1}{2+\frac{r_{10}}{R_0}+2\,\frac{r_{10}}{r_{120}}} \tag{72}$$

Pressure in the recesses at shaft central position P_0 and inlet hydraulic resistance R_0 are not independent. Knowing P_0, the R_0 can be calculated from one of equations (64) or (65) taking into account the flows between recesses at shaft central position are zero. Let us express pressure P_0 through the supply pressure P_s as $P_0 = kP_s$, where k <1. From one of equations (64) or (65) we will get

$$\frac{P_S - k\,P_S}{R_0} = 2\,\frac{k\,P_S}{r_{10}} \qquad \rightarrow \qquad \frac{1}{R_0} = \frac{1}{r_{10}}\,\frac{2\,k}{1-k} \tag{73}$$

Substituting (73) into (72) we are getting for stiffness $C(e = 0)$

$$C(e = 0) = \frac{24\,L_e\,D\,k\,P_S}{\pi\,h_0}\,\frac{1}{\frac{2}{1-k} + 2\,\frac{r_{10}}{r_{120}}} \tag{74}$$

Bu substituting ratio $\frac{r_{10}}{r_{120}}$ from (57) into equation (74) the final formula for the radial stiffness in the shaft central position will be received

$$C(e = 0) = \frac{12\,L_e\,D\,k\,P_S}{\pi\,h_0}\,\frac{1}{\frac{1}{1-k} + \frac{n\,L_e\,L}{\pi\,D\,L_2}} \tag{75}$$

Formula (71) converts to general formula for stiffness (59) in case if pressure in the recess at shaft central position is half of supply pressure and number of recesses is 4.

And the last goal related to the journal bearings will be friction power calculation using the formula (2) from very beginning of tutorial. The surface S of gaps in journal bearing, as can be seen from the bearing drawing, is described by the formula

$$S = 2\,\pi\,D\,L + n\,L_e\,L_2 \tag{76}$$

Substitution (72) in (2) will give a value of power losses in the journal bearing.

But have to be made two important notes.

Note 1. It was neglected contribution from surface of recesses because the depth of recesses is usually many times higher as compared to the radial clearance in the gaps. However, it was shown in very elegant paper written in 1965 by Shinkle, J.N and Hornung, K.G, that circular flow in bearing recesses will increase friction power losses in recesses four times.

Note 2. It was not taken into account a contribution from the turbulence that can be generated at bearing recesses and could has a significant value at very high rotational speeds.